and in the right angled triangles bBO, bfO, the side BO = the side Of, since they are radii of the same circle, and the side bO common, therefore the other angles are equal, that is $ObB = Obf$; also in the right triangles bfO, bfC, the side Of = the side fC by construction and the side bf is common, therefore the angle Obf = the angle Cbf. But Obf was shown to be $= ObB$; therefore $Cbf = ObB$, that is, the three angles are equal to each other. But the three angles Cbf, fbO, ObB, make up the angle $CbB = HOX$ the given angle.

Now in the right angled triangles bfC and OIC the angle bCO is common to the two triangles, therefore the remaining angles are equal, that is, the angle COI = the angle Cbf; but Cbf has been shown to be equal to ObB, therefore $COI = ObB$; but ObB is one-third of CbB or HOX, therefore HOC = one-third HOX.

WM. HILLHOUSE.

New Haven, Conn., Jan., 1878.

SOLUTION OF PROB. 405 BY PROF. C. A. VAN VELZER.—To fix the idea take the determinant of the fourth order

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

To the second column add l_1 times the first, m_1 times the third, and n_1 times the fourth, where l_1, m_1, n_1 are chosen to satisfy the equations

$$(l_1-1)a_1+m_1c_1+n_1d_1+b_1 = 0,$$
$$(l_1-1)a_2+m_1c_2+n_1d_2+b_2 = 0,$$
$$(l_1-1)a_3+m_1c_3+n_1d_3+b_3 = 0.$$

These three equations are sufficient to determine the values of (l_1-1), m_1, n_1, but if to these we add a fourth

$$(l_1-1)a_4+m_1c_4+n_1d_4+b_4 = 0$$

these *four* equations form a consistent set in (l_1-1), m_1, n_1, since the determinant of the coefficients (viz. the original determinant) vanishes.

We see that by this first transformation the determinant reduces to

$$\begin{vmatrix} a_1 & a_1 & c_1 & d_1 \\ a_2 & a_2 & c_2 & d_2 \\ a_3 & a_3 & c_3 & d_3 \\ a_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

Now to the second row of this determinant add l_2 times the first, m_2 times the third and n_2 times the fourth, where l_2, m_2, n_2 satisfy the equ'ns

$$\begin{aligned}(l_2-1)a_1+m_2a_3+n_2a_4+a_2 &= 0,\\ (l_2-1)c_1+m_2c_3+n_2c_4+c_2 &= 0,\\ (l_2-1)d_1+m_2d_3+n_2d_4+d_2 &= 0.\end{aligned}$$

By this transformation the determinant is changed into

$$\begin{vmatrix} a_1 & a_1 & c_1 & d_1 \\ a_1 & a_1 & c_1 & d_1 \\ a_3 & a_3 & c_3 & d_3 \\ a_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

a determinant in which the first two rows are identical and also the first two columns. The same process evidently applies to determinants of any order.

NEW NOTATION FOR ANHARMONIC RATIOS.

BY PROF. WILLIAM WOOLSEY JOHNSON.

1. THE notation here proposed has for its object to express in a symmetrical manner the relation between the six distinct values of the anharmonic ratio of four points and the four modes of writing each which constitute the 24 arrangements of the four letters concerned.

2. The anharmonic ratio of the section of AB by PQ is the ratio

$$\frac{AP}{BP} : \frac{AQ}{BQ};$$

regarding A and B as fixed points of reference, the constituent ratios $AP \div BP$ and $AQ \div BQ$ may be called the *position ratios* of P and Q. A distinction of sign being made between AP and PA, each value of the position ratio determines the position of a point, negative position ratios corresponding to points between A and B; while the position ratio of A is zero and that of B is infinity. The anharmonic ratio is the position ratio of P divided by that of Q, and regarding Q as a third fixed point of reference the value of the anharmonic ratio is a fixed multiple of the position ratio of P, and may be considered a coordinate determining the position of P, in such a manner that the coordinate of Q is unity.

3. Now let this anharmonic ratio be denoted by writing the four letters in a square form thus,

$$\begin{matrix} P & A \\ B & Q \end{matrix} = \frac{AP}{BP} : \frac{AQ}{BQ} = \frac{AP.BQ}{BP.AQ} = x, \tag{1}$$

in which it is to be remembered that the letters occupy the position given to their coordinates in the form

$$\begin{matrix} x & 0 \\ \infty & 1 \end{matrix}$$